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A Numerical Method for Incompressible and Compressible Flow Problems with Smooth Solutions

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A semi-implicit difference method of second order in space is introduced for the numerical solution of the Euler equations. If the Mach number ε is small, the solutions are second-order accurate also in time. In particular, the solutions converge to an approximate solution of the incompressible equations as ε tends to zero. Numerical experiments are presented for channel flow, and the theoretical results (given for the linearized equations) are shown to be valid also for the real nonlinear problem. © 1986 Academic Press, Inc.

1. INTRODUCTION

We consider inviscid compressible flow governed by the barotropic Euler equations, which is a nonlinear hyperbolic system. In particular we shall consider almost incompressible flow, i.e., the Mach number ε is small. In this case the system has time scales of different magnitude, since the sound waves are much faster than the motion of the fluid. If there is little energy in the sound waves, they can be removed from the solution completely without destroying the accuracy. One way of doing this is by using the equations describing incompressible flow, i.e., the equations obtained in the limit as ε tends to zero. Kleinerman, and Majda [11, 12] showed, that with proper initialization the solutions to the equations for compressible flow actually converge to solutions satisfying the equations for incompressible flow.

A general treatment of problems with different time scales has been given by Kreiss [13], Gustafsson [7], Browning and Kreiss [2], Gustafsson and Kreiss [9], and by Tadmor [14]. Special applications have been considered by Browning, Kasahara and Kreiss [3], Gustafsson [8], Barker [1], Ebin [4, 5], and by Kleinerman and Majda in the papers mentioned above.

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In [6] we introduced the leapfrog–backwards Euler difference method for general hyperbolic systems with different time scales. Stability proofs and convergence-rate estimates were given for the case with constant coefficients and periodic solutions. It was also shown that the solutions converge to an approximate solution of the reduced equations, i.e., in our case to the linearized incompressible equations. Here, this method is applied to the nonlinear Euler equations. The geometry chosen is a channel with two-dimensional flow. In this way four different types of boundaries are introduced, namely, solid wall, open inflow boundary, open outflow boundary and a symmetry line.

The numerical method is semi-implicit, which means that part of the differential operator is approximated at the highest time level. This requires the solution of a large system of algebraic equations at each time step.

For nonlinear differential equations this system in general becomes nonlinear, and some iterative procedure is required. For the Euler equations treated here, we split the differential operator in such a way that the implicit part of the approximation is not only linear, but also has constant coefficients. This makes it possible to construct an efficient solution method.

It was shown in [6] that the approximative periodic solutions obtained with the leapfrog/backwards Euler method converge to an approximation of the correct limit solution as ε tends to zero. The basic property that makes this possible is that the implicit part of the difference operator is "large" when ε is small. In the space of periodic grid functions, this can be proved by using Fourier transformations. In this paper we will use a direct technique for the one-dimensional case, to derive necessary restrictions on the boundary conditions, such that the desired property of the implicit operator is retained. Extensive numerical experiments have been performed, and they all show that the method is very robust and produces accurate solutions.

The scheme is only partially dissipative, since the backwards Euler method acts on part of the grid function only. Furthermore there is no damping at all of the highest frequency. Therefore it was expected that extra dissipation terms would have to be added to avoid nonlinear instabilities. However, when using the correct boundary conditions, no signs of instabilities occurred for any of the experiments.

2. The Differential Equations

The Euler equations are

$$\begin{bmatrix} \bar{p} \\ \bar{u} \\ \bar{v} \end{bmatrix}_t + \begin{bmatrix} \bar{u} & \bar{\rho}\bar{a}^2 & 0 \\ 1/\bar{\rho} & \bar{u} & 0 \\ 0 & 0 & \bar{u} \end{bmatrix} \begin{bmatrix} \bar{p} \\ \bar{u} \\ \bar{v} \end{bmatrix}_{\bar{x}} + \begin{bmatrix} \bar{v} & 0 & \bar{\rho}\bar{a}^2 \\ 0 & \bar{v} & 0 \\ 1/\bar{\rho} & 0 & \bar{v} \end{bmatrix} \begin{bmatrix} \bar{p} \\ \bar{u} \\ \bar{v} \end{bmatrix}_{\bar{y}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.1}$$

where $\bar{a} = \sqrt{(d\bar{p}/d\bar{\rho})}$ is the speed of sound. It is assumed that the equation of state is $\bar{p} = A\bar{\rho}^\gamma$. Here $\bar{\rho}$ is the density, $\bar{p}$ is the pressure and $\bar{u}$ and $\bar{v}$ are the x and y velocities, respectively.

We introduce nondimensional variables by

$$x = \frac{\bar{x}}{L}, \qquad y = \frac{\bar{y}}{L}, \qquad t = \frac{\bar{t}}{L/u_0}, \qquad u = \frac{\bar{u}}{u_0}, \qquad v = \frac{\bar{v}}{u_0},$$
$$p = \bar{p}/(\rho_0 u_0^2), \qquad \rho = \bar{\rho}/\rho_0$$

where the subscript 0 indicates a typical value of the corresponding variable. (L is a typical length.) In this way the new system in the new variables is identical to (2.1), except that the equation of state now reads

$$p = \frac{A\rho_0^{\gamma-1}}{u_0^2}\,\rho^\gamma.$$

By introducing the Mach-number $\varepsilon = u_0/a_0$, where $a_0 = (d\bar{p}/d\bar{\rho})_{\bar{\rho}=\rho_0}$ the system (2.1) takes the form

$$\begin{bmatrix} p \\ u \\ v \end{bmatrix}_t + \begin{bmatrix} u & \rho^\gamma/\varepsilon^2 & 0 \\ 1/\rho & u & 0 \\ 0 & 0 & u \end{bmatrix} \begin{bmatrix} p \\ u \\ v \end{bmatrix}_x + \begin{bmatrix} v & 0 & \rho^\gamma/\varepsilon^2 \\ 0 & v & 0 \\ 1/\rho & 0 & v \end{bmatrix} \begin{bmatrix} p \\ u \\ v \end{bmatrix}_y = 0, \tag{2.2}$$

$$p = \frac{\rho^\gamma}{\gamma\varepsilon^2}.$$

When requiring smooth solutions, it is immediately clear from (2.2) that the divergence $u_x + v_y$ must be small.

The system (2.2) is very unsymmetric, and it is not a good basis for a numerical method. Furthermore, the equation of state shows that the pressure tends to infinity when ε tends to zero. To get a more convenient system, we shall symmetrize (2.2) and at the same time eliminate the large part of p.

From the scaled equation of state, we expect that p is a perturbation of $1/(\gamma\varepsilon^2)$. Specifically we take

$$p = \frac{1}{\gamma\varepsilon^2}(1 + \varepsilon c)^{2\gamma/(\gamma-1)}, \tag{2.3}$$

where $c = \tilde{p}(\gamma - 1)/2$. The new "pressure" $\tilde{p}$ is then defined by

$$\tilde{p} = \frac{2\rho^{(\gamma-1)/2}}{\varepsilon(\gamma-1)} - \frac{2}{\varepsilon(\gamma-1)} \tag{2.4}$$

giving the symmetric system,

$$\begin{bmatrix} \tilde{p} \\ u \\ v \end{bmatrix}_t + \begin{bmatrix} u & \frac{1}{\varepsilon}+c & 0 \\ \frac{1}{\varepsilon}+c & u & 0 \\ 0 & 0 & u \end{bmatrix} \begin{bmatrix} \tilde{p} \\ u \\ v \end{bmatrix}_x + \begin{bmatrix} v & 0 & \frac{1}{\varepsilon}+c \\ 0 & v & 0 \\ \frac{1}{\varepsilon}+c & 0 & v \end{bmatrix}_y \begin{bmatrix} \tilde{p} \\ u \\ v \end{bmatrix}_y = 0, \tag{2.5}$$

Kleinerman and Majda [11, 12] analyzed the connection between the compressible and incompressible systems for the Cauchy problem (see also Ebin [5]). They showed that with proper initialization the solution of (2.2) converges as $\varepsilon \to 0$ to the solution of the incompressible system

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= 0, \\ v_t + uv_x + vv_y + p_y &= 0, \\ u_x + u_y &= 0, \end{aligned} \tag{2.6}$$

which is also called the reduced equation.

The value p, as given by (2.3), is not defined for $\varepsilon = 0$. However, only derivatives of p occur in (2.6), and the large part is a constant. Therefore, p in (2.6) should be interpreted as the pressure defined by (2.3), but with the constant $1/(\gamma\varepsilon^2)$ subtracted. By using a Taylor expansion in (2.3), we get

$$p = \frac{1}{\gamma\varepsilon^2} + \frac{\tilde{p}}{\varepsilon} + \mathcal{O}(\tilde{p}^2),$$

showing that $\tilde{p}$ must be of order ε.

This same result can be obtained by applying the Kreiss bounded derivative method [13, 9, 1]. If the time derivatives are bounded initially, they will be bounded on any finite time interval, and we get immediately from (2.5)

$$\begin{gathered} u_x + v_y = \mathcal{O}(\varepsilon), \\ \tilde{p}_x = \mathcal{O}(\varepsilon), \qquad \tilde{p}_y = \mathcal{O}(\varepsilon). \end{gathered}$$

With proper boundary conditions this shows that $\tilde{p} = \mathcal{O}(\varepsilon)$, and that the incompressibility condition $u_x + v_y = 0$ holds in the limit.

By differentiating (2.5), and requiring that the second derivatives are bounded, we obtain in the limit

$$\begin{aligned} u_t + uu_x + vu_y + \hat{p}_x &= 0, \\ v_t + uu_x + vv_y + \hat{p}_y &= 0, \\ \hat{p}_{xx} + \hat{p}_{yy} + 2(u_x^2 + u_y v_x) &= 0, \\ \hat{p} &= \tilde{p}/\varepsilon. \end{aligned} \tag{2.7}$$

This system is often used for computations, since the pressure occurs explicitly in the third equation. It can be derived directly from (2.6) (with $p = \tilde{p}$) by differentiating the first and second equation with respect to x and y, respectively, and adding them.

The numerical method to be presented in this paper is applied directly to the full system (2.4). The structure of this system is very convenient for the application of a semi-implicit method, since the large part of the coefficient matrices is constant. Not only do we avoid the solution of a nonlinear system, but we can also use a constant LU-decomposition and get the solution by simple back-substitutions in each time step. To get a more compact notation, we introduce the vector U and the matrices A_j, B_j by

$$U = \begin{bmatrix} \tilde{p} \\ u \\ v \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} u & c & 0 \\ c & u & 0 \\ 0 & 0 & u \end{bmatrix}, \quad B_2 = \begin{bmatrix} v & 0 & c \\ 0 & v & 0 \\ c & 0 & v \end{bmatrix}.$$

The system to be solved is

$$U_t + \left(\frac{1}{\varepsilon} P_0 + P_1\right) U = 0, \tag{2.8}$$

where

$$\begin{aligned} P_0 &= A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y}, \\ P_1 &= B_1 \frac{\partial}{\partial x} + B_2 \frac{\partial}{\partial y}. \end{aligned}$$

As an application problem we shall consider subsonic flow in a channel according to Fig. 2.1. It is assumed that the flow is symmetric in the y direction, such that the x axis is the symmetry line. The configuration is such that all realistic types of boundaries are present: open inflow and outflow boundaries, solid wall and symmetry line.

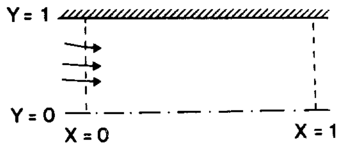

FIG. 2.1. Flow in a channel.

Gustafsson and Kreiss [9] derived well-posed boundary conditions for the open boundaries:

Inflow ($u > 0$ at $x = 0$)

$$\begin{aligned} \alpha \tilde{p} + \varepsilon u &= \varepsilon g_0^{\mathrm{I}}, \qquad \alpha \geqslant 0 \\ v &= g_0^{\mathrm{II}} \end{aligned} \tag{2.9}$$

Outflow ($u > 0$ at $x = 1$)

$$\beta \tilde{p} - \varepsilon u = \varepsilon g_1, \qquad \beta \geqslant 0 \tag{2.10}$$

Here g_0^{I}, g_0^{II}, g_1 are given functions of y and t.

The parameter values $\alpha = 0$ and $\beta = 0$ were not considered in [9]. Well posedness can be proved if one of these parameters is kept nonzero. In such a case the variable $\tilde{p}$ will remain of the order ε, and $\tilde{p}/\varepsilon$ is finite in the limit $\varepsilon = 0$. The *solid wall* boundary condition $v = 0$ is obvious, as well as the *symmetry conditions* $\tilde{p}_y = u_y = v = 0$.

3. The Numerical Method

When solving hyperbolic problems using explicit difference methods, the size of the time step is determined by the fastest propagation speed inherent in the system. If fast waves are present, a small time step must be chosen, and the computing time may become large. In our case we consider almost incompressible flow, i.e., the Mach number is low and the essential variation of solutions is on the slow time scale only. For such a problem accuracy requirements allow for a time step determined by the fluid velocity.

The Crank–Nicholson scheme is unconditionally stable and second order accurate, but it was shown in [6] that nonphysical oscillations occur for small values of ε. The backwards Euler scheme applied on the full system does not have this drawback, but it is only first-order accurate. In the semi-implicit leapfrog–backwards Euler method presented in [6] this latter drawback is also removed. The grid is defined by $(x_i, y_j, t_n) = (i\Delta x, j\Delta y, nk)$, $i = 0, 1, \ldots, M$,

$j=0, 1, \ldots, N$, and the notation U^n_{ij} is used for $U(x_i, y_j, t_n)$. The system (2.4) is approximated by the leapfrog–backwards Euler difference scheme

$$\left(I+\frac{1}{\varepsilon}Q_0\right)U^{n+1}=Q_1(U^n)\,U^n+U^{n-1}, \tag{3.1}$$

where the operators Q_i, $i=0, 1$, are defined as

$$Q_0=2k(A_1D_{0x}+A_2D_{0y}),$$
$$Q_1=-2k(B_1D_{0x}+B_2D_{0y}).$$

Here D_{0x}, D_{0y} are the standard centered difference operators, and the matrices A_i, $B_i=B_i(U^n)$ are defined in Section 2. Initial conditions are such that U^0_{ij} is always given. U^1_{ij} is either determined by the one-step full implicit Backwards Euler scheme

$$\left(I+\frac{1}{2}Q\right)U^1=U^0, \qquad \text{where } Q=\frac{1}{\varepsilon}Q_0-Q_1, \tag{3.2}$$

or it is explicitely given.

The difference scheme (3.1) is applied at inner points, and it requires extra boundary conditions which are not needed in the differential problem. We consider each boundary separately.

The wall, $y=1$. The only boundary condition required by the system of differential equations is $v=0$. The two extra conditions necessary for the difference method are derived from the differential equations. The last equation reduces to $\tilde{p}_y=0$ at $y=1$, and therefore we can equate the pressure at the two upper grid lines. When the second equation is applied at the wall, no y directives occur, hence the centered difference scheme can be applied unmodified. The complete set of boundary conditions at $y=1$ is

$$\begin{aligned}
&\tilde{p}^n_{iN}=\tilde{p}^n_{i,N-1}, && i=1, 2, \ldots, M-1,\\
&u^{n+1}_{i,N}+\frac{2k}{\varepsilon}D_{0x}\,\tilde{p}^{n+1}_{iN}=-u^n_{iN}D_{0x}u^n_{iN}-c^n_{iN}D_{0x}\,\tilde{p}_{iN}+u^{n-1}_{iN}, && \\
& && \mathrm{i}=1, 2, \ldots, M-1,\\
&v^n_{iN}=0, && i=0, 1, \ldots, M
\end{aligned} \tag{3.3}$$

The symmetry line, $y=0$. An extra grid line $\{(x_i, y_{-1}=-\Delta y),\ i=0, 1, \ldots, M\}$ is introduced below the symmetry line, and the difference approximation (3.1) is applied on the symmetry line itself.

The symmetry conditions are

$$\begin{aligned}
&\tilde{p}^n_{i1}=\tilde{p}^n_{i,-1},\\
&u^n_{i1}=u^n_{i,-1},\\
&v^n_{i1}=-v^n_{i,-1}, \qquad i=0, 1, \ldots, M
\end{aligned} \tag{3.4}$$

The inflow boundary, $x=0$. For reasons becoming clear in Section 5, we shall use the value $\alpha=0$ in (2.9), i.e., the velocity is explicitly specified at the inflow boundary. The third condition is obtained by extrapolation of the outgoing characteristic variable $\tilde{p}-u$. We get

$$\begin{aligned} \tilde{p}_{0j}^n - u_{0j}^n &= 2(\tilde{p}_{1j}^n - u_{1j}^n) - (\tilde{p}_{2j}^n - u_{2j}^n), \\ u_{0j}^n &= g_0^I(y_j, t_n), \\ v_{0j}^n &= g_0^{II}(y_j, t_n), \qquad j=0, 1, \ldots, N. \end{aligned} \tag{3.5}$$

The outflow boundary, $x=1$. The variables corresponding to the outgoing characteristics are $\tilde{p}+u$ and v. Considering the one-dimensional problem, the third equation is decoupled from the others, and the approximation reduces to the pure leapfrog scheme. It is well known, see for example [10], that extrapolation along grid points at the same time level leads to an unstable approximation for that scheme. Therefore, the extrapolation for v is modified such that information backwards in time is used. We get

$$\begin{aligned} \tilde{p}_{Mj}^n + u_{Mj}^n &= 2(\tilde{p}_{M-1,j}^n + u_{M-1,j}^n) - (\tilde{p}_{M-2,j}^n + u_{M-2,j}^n), \\ \tilde{p}_{Mj} - \varepsilon u_{Mj} &= \varepsilon g_1(y_j, t_n) \qquad (\beta=1 \text{ in } (2.10)), \\ v_{Mj}^{n+1} &= 2v_{M-1,j}^n - v_{M-2,j}^{n-1}, \qquad j=1, 2, \ldots, N. \end{aligned} \tag{3.6}$$

Note that the given data in the second condition are of order ε, such that the modified pressure also is of order ε in accordance with the arguments of Section 2. In each time-step there is an algebraic system of equations to be solved. Since this system is not only linear but also has constant coefficients, the solution is greatly simplified. The coefficient matrix is LU-decomposed once and for all, and in each time step the solution is obtained by two back-substitutions.

4. Stability

The stability condition for *the Cauchy problem* for approximations of type (3.1) was derived in Theorem 2.2 of [6] under the assumption that the symbols $\hat{Q}_0$, $\hat{Q}_1$ are simultaneously diagonalizable. The latter condition is easily verified in our case, since $\hat{Q}_0$ is skew-symmetric and

$$\hat{Q}_1 = -2\lambda i(u \sin \xi_1 + v \sin \xi_2) I - c\hat{Q}_0, \qquad \lambda = k/h.$$

(Here it is assumed that the space steps both equal h.) The eigenvalues a_j of $\hat{Q}_0/i$ are

$$\begin{aligned} a_1 &= 2\lambda(\sin^2 \xi_1 + \sin^2 \xi_2)^{1/2} \\ a_2 &= -2\lambda(\sin^2 \xi_1 + \sin^2 \xi_2)^{1/2} \\ a_3 &= 0, \end{aligned}$$

with the eigenvalues b_j of $\hat{Q}_1/i$ given by

$$b_j = -2\lambda(u \sin \xi_1 + v \sin \xi_2) - ca_j, \qquad j = 1, 2, 3.$$

Stability holds if for all j one of the conditions

$$a_j = 0, \qquad |b_j| < 2,$$
$$|a_j/\varepsilon| > |b_j|,$$

is satisfied. For small values of ε, the only nontrivial condition is $|b_3| < 2$, which implies the final stability condition

$$k < \frac{h}{|u| + |v|}. \tag{4.1}$$

Next we shall consider *the mixed initial boundary value problem.* We will limit ourselves to the one-dimensional case, the theoretical results derived are later verified for the two-dimensional problem by numerical experiments. The approximation is

$$U_j^{n+1} + \frac{\lambda}{\varepsilon} A_1(U_{j+1}^{n+1} - U_{j-1}^{n+1}) + \lambda B_1(U_{j+1}^n - U_{j-1}^n) = U_j^{n-1},$$

where A_1, B_1 are defined in Section 2. The coefficient matrices can be simultaneously diagonalized, giving the new system

$$\begin{gathered} V_j^{n+1} + \frac{\lambda}{\varepsilon} \bar{A}(V_{j+1}^{n+1} - V_{j-1}^{n+1}) + \lambda \bar{B}(V_{j+1}^n - V_{j-1}^n) = V_j^{n-1}, \\ V_j = T^{-1} U_j = (v_j^{(1)}, v_j^{(2)}, v_j^{(3)})^T \end{gathered} \tag{4.2}$$

$$T = (1/\sqrt{2}) \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} u+c & 0 & 0 \\ 0 & u-c & 0 \\ 0 & 0 & u \end{bmatrix}.$$

The boundary conditions (3.5) and (3.6) written in homogeneous form becomes (all variables evaluated at time level n unless otherwise specified)

$$\begin{aligned} v_0^{(2)} - 2v_1^{(2)} + v_2^{(2)} &= 0, \\ v_0^{(1)} - v_0^{(2)} &= 0, \\ v_0^{(3)} &= 0, \end{aligned} \tag{4.3}$$

at the inflow boundary, and

$$v_M^{(1)} - 2v_{M-1}^{(1)} + v_{M-2}^{(1)} = 0,$$
$$\frac{1}{\varepsilon}(v_M^{(1)} + v_M^{(2)}) - (v_M^{(1)} - v_M^{(2)}) = 0, \tag{4.4}$$
$$v_M^{(3)^{n+1}} = 2v_{M-1}^{(3)^n} - v_{M-2}^{(3)^{n-1}},$$

at the outflow boundary. The second outflow condition has been normalized such that it corresponds to data of order one.

The normal mode analysis leads to the determinant condition, see [10], which is derived for the coefficients of the solution to the resolvent equation. With proper normalization of the variables, it means that the determinant of the system is different from zero uniformly in z for $|z| > 1$.

Since we are dealing with the small parameter ε in our system, it is natural to strengthen the stability condition, such that the estimates of the solution do not break down as ε approaches zero. Therefore we shall call the approximation *uniformly stable* if the determinant condition is satisfied uniformly in ε. By this we mean that the lower bound on the magnitude of the determinant is independent of ε as $\varepsilon \to 0$.

We shall first consider the quarter-space problems, where the equations are defined either in the domain $\{0 \leqslant x,\ 0 \leqslant t\}$ or in the domain $\{x \leqslant 1,\ 0 \leqslant t\}$.

THEOREM 4.1. *The leapfrog–backwards Euler scheme is uniformly stable with the boundary conditions* (4.3) *for the right quarter space problem, and with* (4.4) *for the left quarter space problem.*

Proof. The third equation is decoupled from the others, and since it is independent of ε, uniform stability is trivial when $v_0^{(3)}$ is specified. The first two components $\hat{v}^{(1)}$, $\hat{v}^{(2)}$ of the solutions to the resolvent equation have the form (for the right quarter space problem)

$$\hat{v}_j^{(1)} = \sigma\kappa^j, \qquad |\kappa| < 1,$$
$$\hat{v}_j^{(2)} = \tau\mu^j, \qquad |\mu| < 1,$$

where the characteristic equations for κ and μ are

$$(z^2 - 1)\,\kappa + \lambda z(u + c)(\kappa^2 - 1) + \frac{\lambda}{\varepsilon} z^2(\kappa^2 - 1) = 0, \tag{4.5a}$$

$$(z^2 - 1)\,\mu + \lambda z(u - c)(\mu^2 - 1) - \frac{\lambda}{\varepsilon} z^2(\mu^2 - 1) = 0. \tag{4.5b}$$

The boundary conditions give the relations

$$\sigma - \tau = 0,$$
$$(\mu - 1)^2 \tau = 0,$$

and the determinant is $(\mu - 1)^2$. Obviously, we only have to prove that μ never approaches 1 for any z, $|z| > 1$, and for any ε, $0 < \varepsilon \ll 1$. From (4.5b) we get

$$\mu^2 - 1 + \frac{\varepsilon(z^2 - 1)}{\lambda z[\varepsilon(u - c) - z]} \mu = 0.$$

The coefficient for μ is of the order ε even if $|z|$ is large. Hence the roots μ_1, μ_2 of (4.5b) satisfy

$$\mu_1 + \mu_2 = \mathcal{O}(\varepsilon),$$
$$\mu_1 \mu_2 = -1,$$

yielding

$$\begin{aligned} \mu_1 &= \pm 1 + \mathcal{O}(\varepsilon), \qquad |\mu_1| < 1 \text{ for } |z| > 1, \\ \mu_2 &= \pm 1 + \mathcal{O}(\varepsilon), \qquad |\mu_2| > 1 \text{ for } |z| > 1. \end{aligned} \tag{4.6}$$

It remains to show that only the minus sign is valid for μ_1. For $\mu = 1$ we must have $z = \pm 1$, and we introduce a perturbation for z, and set $z = 1 + \delta_1$, $\delta_1 > 0$, $\mu = 1 + \delta_2$ in (4.5b). Keeping only first order terms in δ_1, δ_2 gives

$$\delta_2 = \frac{\delta_1}{-\lambda(u - c) + \lambda/\varepsilon} > 0$$

showing that $\mu = \mu_2$. Similarly for $z = -(1 + \delta_1)$, $\delta_1 > 0$, $\mu = 1 + \delta_2$, we get

$$\delta_2 = \frac{\delta_1}{-\lambda(u - c) + \lambda/\varepsilon} > 0$$

again showing that $\mu = \mu_2$, and stability follows for the right quarter space problem. For the boundary $x = 1$, the same analysis holds almost unchanged. The outgoing characteristic variable is also here extrapolated, and the only difference is the coupling between the variables $v^{(1)}$, $v^{(2)}$. For the resolvent variables we have

$$\hat{v}_M^{(2)} = -\frac{1 - \varepsilon}{1 + \varepsilon} \hat{v}_M^{(1)}$$

and obviously no extra difficulty occurs for $\varepsilon \to 0$. The extrapolation used for $v^{(3)}$ is known to be stable, see [10], and the parameter ε is not involved. This proves the theorem.

The general theory for mixed initial-boundary value problems shows that stability follows for the strip problem with two boundaries if the quarter-space problems are stable. However, in general we cannot expect that the same property is valid also for uniform stability. This is most easily understood by considering the differential equation. Normally the characteristics travel across the domain with finite speed of propagation, hence the right boundary "knows" what happened at the left boundary only after a time period of order one. For our system however, this reaction time is of order ε, which means that there is a much closer coupling between the boundaries. We shall prove that uniform stability holds for our approximation.

THEOREM 4.2. *The leapfrog–backwards Euler scheme is uniformly stable for the strip problem with the boundary conditions* (4.3), (4.4).

Proof. The equation for $v^{(3)}$ is trivial, since it is independent of ε. The general solution to the resolvent equation for $v^{(1)}$, $v^{(2)}$ has the form

$$v_j^{(1)} = \sigma_1 \kappa_1^j + \sigma_2 \kappa_2^{j-M}, \qquad |\kappa_1| < 1,\ |\kappa_2| > 1,$$
$$v_j^{(2)} = \tau_1 \mu_1^j + \tau_2 \mu_2^{j-M}, \qquad |\mu_1| < 1,\ |\mu_2| > 1,$$

where κ_1, κ_2 and μ_1, μ_2 are the roots to the characteristic equations (4.5a) and (4.5b), respectively. The relations (4.6) given for μ_1, μ_2 obviously hold also for κ_1, κ_2. The boundary conditions give

$$\begin{aligned}
\sigma_1 + \sigma_2 \kappa_2^{-M} - \tau_1 - \tau_2 \mu_2^{-M} &= 0,\\
\tau_1(\mu_1 - 1)^2 + \tau_2(\mu_2 - 1)^2 \mu_2^{-M} &= 0,\\
-\left(1 - \frac{1}{\varepsilon}\right)(\sigma_1 \kappa_1^M + \sigma_2) + \left(1 + \frac{1}{\varepsilon}\right)(\tau_1 \mu_1^M + \tau_2) &= 0,\\
\sigma_1 \kappa_1^{M-2}(\kappa_1 - 1)^2 + \sigma_2 \kappa_2^{-2}(\kappa_2 - 1)^2 &= 0.
\end{aligned}$$

We denote by Q the corresponding coefficient matrix

$$Q = \begin{vmatrix}
1 & \kappa_2^{-M} & -1 & -\mu_2^{-M} \\
0 & 0 & (\mu_1 - 1)^2 & (\mu_2 - 1)^2 \mu_2^{-M} \\
-\left(1 - \frac{1}{\varepsilon}\right)\kappa_1^M & -\left(1 - \frac{1}{\varepsilon}\right) & \left(1 + \frac{1}{\varepsilon}\right)\mu_1^M & 1 + \frac{1}{\varepsilon} \\
\kappa_1^{M-2}(\kappa_1 - 1)^2 & \kappa_2^{-2}(\kappa_2 - 1)^2 & 0 & 0
\end{vmatrix}.$$

We have

$$\operatorname{Det} Q = -\left(\mu_1 - 1)^2 \left\{\kappa_1^{M-2}(\kappa_1 - 1)^2 \left[\left(1 + \frac{1}{\varepsilon}\right)\kappa_2^{-M} - \left(1 - \frac{1}{\varepsilon}\right)\mu_2^{-M}\right]\right.\right.$$
$$\left. - \kappa_2^{-2}(\kappa_2 - 1)^2 \left[\left(1 + \frac{1}{\varepsilon}\right) - \left(1 - \frac{1}{\varepsilon}\right)\mu_2^{-M}\kappa_1^{M}\right]\right\}$$
$$+ (\mu_2 - 1)^2 \mu_2^{-M} \left\{\kappa_1^{M-2}(\kappa_1 - 1)^2 \left[\left(1 + \frac{1}{\varepsilon}\right)\mu_1^M \kappa_2^{-M} - \left(1 - \frac{1}{\varepsilon}\right)\right]\right.$$
$$\left. - \kappa_2^{-2}(\kappa_2 - 1)^2 \left[\left(1 + \frac{1}{\varepsilon}\right)\mu_1^M - \left(1 - \frac{1}{\varepsilon}\right)\kappa_1^M\right]\right\}.$$

It follows that

$$\operatorname{Det} Q = [(\mu_2 - 1)^2 - (\mu_1 - 1)^2]\left\{\left(\kappa_1^{M-2}(\kappa_1 - 1)^2 \left[\left(1 + \frac{1}{\varepsilon}\right)\kappa_2^{-M} - \mu_2^{-M}\left(1 - \frac{1}{\varepsilon}\right)\right]\right\}\right.$$
$$\left. - \kappa_2^{-2}(\kappa_2 - 1)^2 \left[\left(1 + \frac{1}{\varepsilon}\right) - \left(1 - \frac{1}{\varepsilon}\right)\mu_2^{-M}\mu_1^M\right]\right\}$$
$$+ (\mu_2 - 1)^2 \left(1 + \frac{1}{\varepsilon}\right)\left(\left(\frac{\mu_1}{\mu_2}\right)^M - 1\right)\left[\left(\frac{\kappa_1}{\kappa_2}\right)^M (\kappa_1 - 1)^2 \kappa_1^{-2} - \kappa_2^{-2}(\kappa_2 - 1)^2\right].$$

Since each quarter space problem is stable, we know that

$$\kappa_1 \cong 1, \qquad \kappa_2 \cong -1,$$
$$\mu_1 \cong -1, \qquad \mu_2 \cong 1.$$

According to (4.6) and the analogous relations for κ_1, κ_2, we set

$$\kappa_1 = \exp(-\alpha_1 \varepsilon), \qquad \kappa_2 = -\exp(\alpha_2 \varepsilon),$$
$$\mu_1 = -\exp(-\beta_1 \varepsilon), \qquad \mu_2 = \exp(\beta_2 \varepsilon),$$

where α_1, α_2, β_1, $\beta_2 > 0$.

Thus

$$\operatorname{Det} Q = -4\left[(\alpha_1 \varepsilon)^2 \left\{(-1)^M \exp[-(\alpha_1 + \alpha_2)\varepsilon M]\frac{1}{\varepsilon} + \exp[-(\alpha_2 + \beta_2)\varepsilon M]\frac{1}{\varepsilon}\right\}\right.$$
$$\left. - 4\left\{\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\exp[-(\alpha_1 + \beta_2)\varepsilon M]\right\}\right]$$
$$+ (\beta_2 \varepsilon)^2 \frac{1}{\varepsilon}((-1)^M \exp[-(\beta_1 + \beta_2)\varepsilon M] - 1)$$
$$\times \{(\alpha_1 \varepsilon)^2 (-1)^M \exp[-(\alpha_1 + \alpha_2)\varepsilon M] - 4\}.$$

Apparently, Det Q is of order $1/\varepsilon$. This indicates that we have not only uniform stability as $\varepsilon \to 0$, but we also gain a factor ε if non-zero data are introduced. In the next section we shall show that this is actually the case.

5. Accuracy

It was shown in [6] that the leapfrog–backwards Euler method for periodic problems with constant coefficients give approximations to smooth solutions with second-order accuracy if ε is small enough. More precisely, the estimate has the form

$$\| U(t_n) - U^n \| \leqslant C(h^2 + \varepsilon k). \tag{5.1}$$

A closer look at the proof reveals that the extra factor ε, which improves the first-order accuracy and which also provides convergence to the reduced equation, is obtained because the operator $(I + Q_0/\varepsilon)^{-1}$ is of order ε/k (expect for the lowest and the highest frequency). This fact is in that case easily established by using a Fourier transformation. In the present case with boundaries involved, the situation is more complicated.

We will again limit ourselves to the one-dimensional case which is sufficient to illustrate the influence of the boundary conditions on the accuracy. We shall study the operator

$$\frac{1}{\varepsilon} Q_0 \equiv \frac{2k}{\varepsilon} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} D_0 \tag{5.2}$$

acting on one-dimensional vectors

$$U_j = \begin{bmatrix} \tilde{p}_j \\ u_j \end{bmatrix}.$$

We shall consider the homogeneous boundary conditions

$$\begin{aligned}
&\text{(a)} && u_0 + \frac{\alpha}{\varepsilon} \tilde{p}_0 = 0, \\
&\text{(b)} && u_M - \frac{\beta}{\varepsilon} \tilde{p}_M = 0, \\
&\text{(c)} && u_0 - \tilde{p}_0 - 2(u_1 - \tilde{p}_1) + u_2 - \tilde{p}_2 = 0, \\
&\text{(d)} && u_M + \tilde{p}_M - 2(u_{M-1} + \tilde{p}_{M-1}) + u_{M-2} + \tilde{p}_{M-2} = 0.
\end{aligned} \tag{5.3}$$

The differential equations are well posed with the conditions (5.3a), (5.3b) for $\alpha \geqslant 0$, $\beta \geqslant 0$ for both compressible and incompressible flow. We shall prove that a stronger restriction is required for the difference approximation in order to obtain the right order of accuracy.

LEMMA 5.1. *Consider the difference operator* $(I+Q_0/\varepsilon)$, *where* Q_0 *is defined by* (5.2), *acting on grid functions* U_j *satisfying the boundary conditions* (5.3). *If* $\varepsilon = o(k)$ (*this assumption can be removed at the expense of a more elaborate proof*), *then the estimate*

$$\|(I+Q_0/\varepsilon)^{-1}\| \leqslant C\frac{\varepsilon}{k} \tag{5.4}$$

holds if and only if $\alpha = 0$. (*It is assumed that* α *is independent of* ε.) *Here,* $\|\cdot\|$ *denotes the norm corresponding to the maximum norm for vectors.*

Proof. We consider the equation

$$Q_0\begin{bmatrix}\tilde{p}\\ u\end{bmatrix} = \begin{bmatrix}F\\ G\end{bmatrix} \tag{5.5}$$

with boundary conditions (5.3). The case M *even* is first treated. Let $\sum_o$, $\sum_e$ denote summation over odd and even integers, respectively. The solution is for even j

$$u_j = u_0 + \lambda \sum_{\nu=0}^{j-1}{}_o F_\nu, \qquad j = 2, 4, \ldots, M,$$

$$\tilde{p}_j = \tilde{p}_0 + \lambda \sum_{\nu=1}^{j-1}{}_o G_\nu, \qquad j = 2, 4, \ldots, M.$$

In particular we have

$$u_M = u_0 + \lambda \sum_{\nu=1}^{M-1}{}_o F_\nu = -\frac{\alpha}{\varepsilon}\tilde{p}_0 + \lambda \sum_{\nu=1}^{M-1}{}_o F_\nu$$

and

$$\frac{\varepsilon u_M}{\beta} = \tilde{p}_M = \tilde{p}_0 + \lambda \sum_{\nu=1}^{M-1}{}_o G_\nu.$$

$\tilde{p}_0$ can be eliminated, and we get

$$u_M + \frac{\alpha}{\beta}u_M = \lambda \sum_{\nu=1}^{M-1}{}_o \left(F_\nu + \frac{\alpha}{\varepsilon}G_\nu\right),$$

which gives

$$u_M = \left(1+\frac{\alpha}{\beta}\right)^{-1} \lambda \sum_{\nu=1}^{M-1}{}_o \left(F_\nu + \frac{\alpha}{\varepsilon}G_\nu\right).$$

Obviously, the estimate (5.4) holds only if $|u_M|$ is bounded for small values of ε, hence we must have $\alpha = 0$. Next assume that α is zero, i.e., the velocity is explicitly given at the inflow boundary. We have

$$|u_M| \leqslant \lambda M \, \|F\| = \frac{\lambda^2}{k} \|F\|,$$

and by using the relations between u_0, u_j, u_M given above, we get

$$|u_j| \leqslant \frac{C_1}{k} \|F\|, \qquad j = 0, 2, ..., M. \tag{5.6a}$$

Similarly we get for $\tilde{p}$

$$|\tilde{p}_j| \leqslant \frac{C_2}{k} (\|G\| + \varepsilon \|F\|), \qquad j = 0, 2, ..., M. \tag{5.6b}$$

The values of $\tilde{p}$ and u at odd grid points are coupled to the even points by the extrapolation conditions (5.3c), (5.3d). Since the parameter ε is not present in these equations, the estimates for odd points follows immediately from (5.6a), (5.6b). The final estimate is

$$\max(\|\tilde{p}\|, \|u\|) \leqslant \frac{C}{k} (\|F\| + \|G\|). \tag{5.7}$$

Next we consider M *odd*, and get

$$u_j = u_1 + \lambda \sum_{\nu=2}^{j-1}{}_e F_\nu, \qquad j = 3, 5, ..., M,$$

$$\tilde{p}_j = \tilde{p}_1 + \lambda \sum_{\nu=2}^{j-1}{}_e G_\nu, \qquad j = 3, 5, ..., M.$$

When setting $j = M-2$, M and adding these relations, we obtain after using (5.3d)

$$u_{M-1} + \tilde{p}_{M-1} = \frac{1}{2}(u_M + \tilde{p}_M + u_{M-2} + \tilde{p}_{M-2}) = u_1 + \tilde{p}_1 + \lambda \sum_{\nu=2}^{M-1}{}_e (F_\nu + G_\nu)$$
$$- \frac{\lambda}{2}(F_{M-1} + G_{M-1}).$$

We also have

$$u_{M-1} = u_0 + \lambda \sum_{\nu=1}^{M-2}{}_o F_\nu,$$

$$\tilde{p}_{M-1} = \tilde{p}_0 + \lambda \sum_{\nu=1}^{M-2}{}_o G_\nu,$$

which gives

$$u_0+\tilde{p}_0=u_1+\tilde{p}_1+\lambda\left[\sum_{\nu=2}^{M-1}{}_e\,(F_\nu+G_\nu)-\sum_{\nu=1}^{M-2}{}_o\,(F_\nu+G_\nu)\right]$$
$$-\frac{\lambda}{2}(F_{M-1}+G_{M-1}) \tag{5.8}$$

The difference equations give

$$u_2=u_0+\lambda F_1,$$
$$\tilde{p}_2=\tilde{p}_0+\lambda G_1,$$

and after substitution into the boundary condition (5.3c) we get

$$u_0-\tilde{p}_0=u_1-\tilde{p}_1+\frac{\lambda}{2}(G_1-F_1). \tag{5.9}$$

When adding the equations (5.8) and (5.9) we obtain

$$u_0=u_1+\frac{\lambda}{4}(G_1-F_1-F_{M-1}-G_{M-1})+\frac{\lambda}{2}\left[\sum_{\nu=2}^{M-1}{}_e\,(F_\nu+G_\nu)-\sum_{\nu=1}^{M-2}{}_o\,(F_\nu+G_\nu)\right],$$

and after subtraction of (5.9) from (5.8)

$$\tilde{p}_0=\tilde{p}_1+\frac{\lambda}{4}(F_1-G_1-F_{M-1}-G_{M-1})+\frac{\lambda}{2}\left[\sum_{\nu=2}^{M-2}{}_e\,(F_\nu+G_\nu)-\sum_{\nu=1}^{M-2}{}_o\,(F_\nu+G_\nu)\right]$$

We also have

$$u_M=u_1+\lambda\sum_{\nu=2}^{M-1}{}_e\,F_\nu=u_0+\lambda\left[\sum_{\nu=2}^{M-2}{}_e\left(F_\nu-\frac{1}{2}F_\nu-\frac{1}{2}G_\nu\right)\right.$$
$$\left.+\frac{1}{2}\sum_{\nu=1}^{M-2}{}_o\,(F_\nu+G_\nu)+\frac{1}{4}(F_1-G_1+F_{M-1}+G_{M-1})\right]$$

$$\frac{\varepsilon}{\beta}u_M=\tilde{p}_M=\tilde{p}_1+\lambda\sum_{\nu=2}^{M-1}{}_e\,G_\nu=\tilde{p}_0+\lambda\left[\sum_{\nu=2}^{M-1}{}_e\left(G_\nu-\frac{1}{2}F_\nu-\frac{1}{2}G_\nu\right)\right.$$
$$\left.+\frac{1}{2}\sum_{\nu=1}^{M-2}{}_o\,(F_\nu+G_\nu)+\frac{1}{4}(G_1-F_1+F_{M-1}+G_{M-1})\right]$$

When multiplying the last equation by α/ε and adding it to the previous one, we get

$$\left(1+\frac{\alpha}{\beta}\right)u_M=\frac{1}{k}\left(T_1+\frac{\alpha}{\varepsilon}T_2\right),$$

where T_1, T_2 are independent of ε. Hence, it is necessary that $\alpha = 0$ also for odd M. Under this condition the estimate (5.7) is obtained in the same way as for the case M even.

We have proved that kQ_0^{-1} is bounded, or equivalently that

$$\| Q_0 U \| \geqslant Ck \| U \|, \qquad C > 0. \tag{5.10}$$

But then

$$\left\| \left(I + \frac{1}{\varepsilon} Q_0 \right) U \right\| \geqslant \left| \frac{1}{\varepsilon} \| Q_0 U \| - \| U \| \right| \geqslant \left(\frac{Ck}{\varepsilon} - 1 \right) \| u \| \geqslant \frac{C_1 k}{\varepsilon} \| U \|$$

and (5.4) follows immediately.

Lemma 5.1 can also be applied to the difference operator acting in the y direction. In that case, the boundary conditions are independent of ε. Therefore the estimate (5.10) follows immediately and (5.4) holds. When deriving the complete error estimate, the truncation error must be entered into the right-hand side of the boundary condition. In (5.3c), (5.3d) these terms are of order h^2, and there is no loss of accuracy. In the y direction, there is the numerical boundary condition $\tilde{p}_{i,N} = \tilde{p}_{i,N-1}$. This is normally only first-order accurate, but in our case the solutions satisfy $\tilde{p}_y(x, 1, t) = 0$ which gives

$$\tilde{p}(x, 1, t) = \tilde{p}(x, 1-h, t) + \frac{h^2}{2} \tilde{p}_{yy}(x, 1, t) + \mathcal{O}(h^3)$$

showing second-order accuracy.

6. Numerical Test-Runs

An extensive number of test-runs have been performed, which all confirm the theoretical (but sometimes simplified) analysis given in this paper. These numerical experiments will be published elsewhere. Here we shall only show the most important properties, namely that the solutions converge to accurate solutions as ε tends to zero, and that the scheme is actually giving "incompressible solutions" even if the data are perturbed.

The first experiment simulates a case where the fluid is initially at rest, and then put into motion by increasing the velocity smoothly at the left boundary. The boundary conditions are

$x = 0$

$$u = \begin{cases} 0.6 \left[\sin^2 \dfrac{\pi t}{2} + \sin^2(\pi t) \sin^2(\pi y) \right], & 0 \leqslant t \leqslant 1 \\[2ex] 0.6, & 1 \leqslant t \end{cases}$$

$$v = 0.$$

$x = 1$

$$\tilde{p} - \varepsilon u = 0.$$

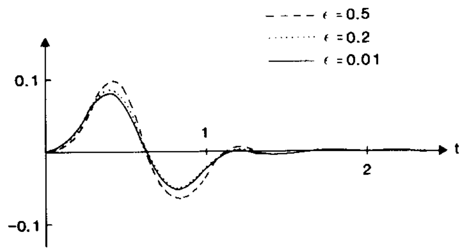

FIG. 6.1. The velocity component v at $x=0.05$, $y=0.40$.

Figure 6.1 shows the result for $\varepsilon=0.5$, 0.2, 0.01, with step-sizes $h=0.05$, $k=0.025$. The convergence is clearly illustrated. As a measure of incompressibility the quantity

$$\mathrm{DIV}=\left(\sum_{i,j}|D_{+x}u_{ij}+D_{+y}v_{ij}|^2\,h^2\right)^{1/2} \tag{6.1}$$

was computed, being less than 10^{-3} in all cases. In the second experiment a strong perturbation was introduced initially, such that the data are no longer divergence-free. The initial data were

$$u=0.1+\cos(2\pi x),$$
$$v=0,$$
$$\tilde{p}=0.$$

Figure 6.2 shows how the full backwards Euler first step takes the divergence down such that the solution goes back to the "incompressible track." When the semi-implicit scheme is applied at the second step, a slight increase of the divergence occurs, but it is later damped out again. Since the backwards Euler

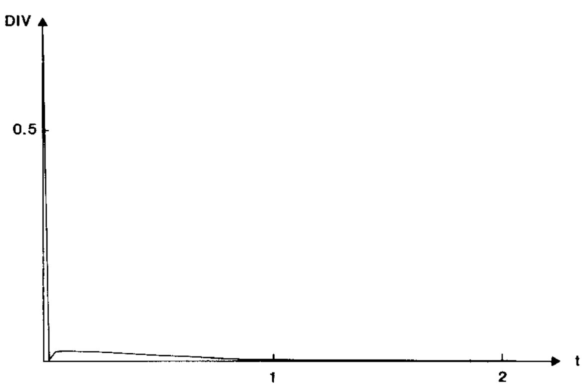

FIG. 6.2. DIV defined in (6.1) as a function of time.

TABLE I

h \ ε	1	0.1	0.01
0.2	0.2735	0.5409	0.53988
0.1	0.2627	0.5434	0.54002
0.05	0.2571	0.5442	0.53999

method is applied on part of the system only, and since it is not strictly dissipative even for those components where it is applied, there seems to be good reason for adding extra dissipation terms. However, the method have shown a very robust behaviour, and there have been no signs of nonlinear instabilities even for large time-intervals.

The accuracy was tested by running the scheme for $h = 0.2$, 0.1, 0.05. The values of u at an inner point are given in Table I for three different ε-values. The error estimate has the form (5.1) and we expect the (εk) term to dominate for larger values of ε. Since k is chosen proportional to h, we make the ansatz

$$\tilde{U}(\varepsilon, h) - U(\varepsilon) \approx c_1(\varepsilon)\, \varepsilon h + c_2(\varepsilon)\, h^2, \tag{6.2}$$

where $\tilde{U}$ is the computed solution.

If h is small enough, the first error term always dominates. However, for practical calculations, the second error term is the largest for small values of ε. Assuming that the error is proportional to $h^{q(\varepsilon)}$, we get

$$q(\varepsilon) \approx \log \left[\frac{U(\varepsilon, h/2) - U(\varepsilon, h)}{U(\varepsilon, h/4) - U(\varepsilon, h)} \right] \Big/ \log 2. \tag{6.3}$$

When moving from left to right in Table I, the quantity on the right-hand side of (6.3) should go from 1 to 2. We get from our experiment $q(1) = 0.9$, $q(0.1) = 1.7$. The third column shows that the calculation is at the level of round-off, i.e., the error is of the order ϕ_m/k, where ϕ_m is the machine precision (IBM single precision was used).

7. Conclusion

The leapfrog–backwards Euler method that was developed and analyzed for periodic problems in [6] has been applied to subsonic flow problems at various Mach numbers ε. A modified pressure is defined such that the system becomes symmetric and well conditioned even when ε is very small. It has been shown that the method is accurate for almost incompressible flow, and the solutions converge as the Mach number goes to zero. Accordingly, the method provides a simple and efficient way of computing also incompressible flow; the divergence-free property is automatically taken care of.

In each step an algebraic system of equations must be solved, but it is linear and has constant coefficients. In this way a direct solver can be used even for large problems, since each step requires only two backsubstitutions.

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